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An Analysis of Cardinal Spline-Wavelets*

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The mth order cardinal B-spline-wavelet (or simply, B-wavelet) ψ_m is known to generate orthogonal decompositions of any function in $L^2(-\infty, \infty)$. Since ψ_m is usually considered as a bandpass filter, a wavelet series $g = \sum c_j \psi_m(\cdot - j)$ may be treated as a bandpass signal. Hence, the problem of characterizing g from its "zero-crossings" is very important in the application of spline-wavelets to signal analysis. However, since g is not an entire function. weak sign changes of g must also be taken into consideration. The objective of this paper is to initiate a study of this important problem. It is noted, in particular, that in contrast to the total positivity property of the mth order B-spline, the B-wavelet ψ_m seems to possess a remarkable property, which we call "complete oscillation." © 1993 Academic Press, Inc.

1. INTRODUCTION

For any positive integer m, let N_m denote the mth order cardinal B-spline with integer knot sequence $\mathbb{Z}$, defined recursively by

$$N_m(x) := (N_{m-1} * N_1)(x) = \int_0^1 N_{m-1}(x-t)\,dt, \tag{1.1}$$

where $N_1 = \chi_{(0,1]}$ is the characteristic function of the unit interval (0, 1]. Corresponding to the N_m, the mth order cardinal B-spline-wavelet ψ_m (which will be called "B-wavelet" for short), introduced in our earlier work [3], is defined by

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$$\psi_m(x) := 2^{-m+1} \sum_{j=0}^{2m-2} (-1)^j N_{2m}(j+1) N_{2m}^{(m)}(2x-j)$$
$$= \sum_{j=0}^{3m-2} q_{m,j} N_m(2x-j), \tag{1.2}$$

where

$$q_{m,j} = \frac{(-1)^j}{2^{m-1}} \sum_{l=0}^{m} \binom{m}{l} N_{2m}(j-l+1). \tag{1.3}$$

While it is well known (cf. [10, 12]) that the B-spline N_m generates a multiresolution analysis of $L^2 := L^2(-\infty, \infty)$,

$$\cdots \subset V_{-1}^m \subset V_0^m \subset V_1^m \subset \cdots$$

in the sense that

$$V_k^m = \operatorname{clos}_{L^2} \langle N_m(2^k \cdot -j) \colon j \in \mathbb{Z} \rangle$$

for all $k \in \mathbb{Z}$, it was shown in [3] that the B-wavelet ψ_m generates the orthogonal subspaces W_k^m in the sense that

$$W_k^m = \operatorname{clos}_{L^2} \langle \psi_m(2^k \cdot -j) \colon j \in \mathbb{Z} \rangle$$

for all $k \in \mathbb{Z}$. Here, we have, for each $k \in \mathbb{Z}$,

$$V_{k+1}^m = V_k^m + W_k^m, \qquad V_k^m \perp W_k^m;$$

and the notation

$$V_{k+1}^m = V_k^m \oplus W_k^m$$

for the orthogonal summation will be used. Hence, it follows from the facts

$$\operatorname{clos}_{L^2}\left(\bigcup_{k \in \mathbb{Z}} V_k^m \right) = L^2 \qquad \text{and} \qquad \bigcap_{k \in \mathbb{Z}} V_k^m = \{0\}$$

that

$$L^2 = \bigoplus_{k \in \mathbb{Z}} W_k^m := \cdots \oplus W_{-1}^m \oplus W_0^m \oplus W_1^m \oplus \cdots.$$

In other words, every function $f \in L^2$ has a (unique) orthogonal wavelet decomposition:

$$f = \sum_{k \in \mathbb{Z}} g_k, \qquad g_k \in W_k^m;$$
$$g_k = \sum_{j \in \mathbb{Z}} d_j^k \psi_m(2^k \cdot -j). \tag{1.4}$$

The significance of the coefficients d_j^k in the wavelet series decomposition (1.4) of f is that they are constant multiples of the integral wavelet transform

$$(W_{\tilde{\psi}_m} f)(b, a) = \frac{1}{\sqrt{a}} \int_{-\infty}^{\infty} f(t) \tilde{\psi}_m \left(\frac{t-b}{a} \right) dt \tag{1.5}$$

of f relative to the dual wavelet $\tilde{\psi}_m$ of ψ_m at the dyadic time-scale locations

$$b = \frac{j}{2^k}, \qquad a = \frac{1}{2^k}.$$

More precisely, $\tilde{\psi}_m \in W_0^m$ is defined by

$$\int_{-\infty}^{\infty} \psi_m(t) \tilde{\psi}_m(t-j)\, dt = \delta_{j,0}, \qquad j \in \mathbb{Z},$$

and

$$d_j^k = 2^{k/2} (W_{\tilde{\psi}} f) \left(\frac{j}{2^k}, \frac{1}{2^k} \right). \tag{1.6}$$

For more details, the reader is referred to [3]. The objective of this paper, on the other hand, is to analyze the wavelet series

$$g_k = \sum_{j \in \mathbb{Z}} d_j^k \psi_m(2^k \cdot -j),$$

which constitutes the kth component of the orthogonal wavelet decomposition of f.

In signal analysis, a bandpass (and band-limited) signal u is an entire function of the exponential type with

$$\operatorname{supp} |\hat{u}| \subset [-\beta, -\alpha] \cup [\alpha, \beta]$$

for some α and β, where $0 < \alpha < \beta < \infty$. It is well known (cf. [9]) that the distribution of the zeros of such a signal u is governed by the "Nyquist rate" (cf. [13, p. 87]), and under certain conditions, u can be completely recovered from its isolated sign changes (or better known as "zero-crossings"). In the application of wavelets to signal processing, any wavelet, such as ψ_m, is also considered as a bandpass filter (cf. [5, 6, 10]). Consequently, the wavelet series g_k in (1.7) may also be treated as a bandpass signal. However, since g_k is only a spline function, it is not a band-limited signal. Hence. a very important problem is to study the behavior of the isolated sign changes of g_k, and, particularly, conditions under which the wavelet series g_k is characterized by its isolated sign changes.

This paper is intended to initiate a study of the above stated problem. It will be observed that in contrast to the "total positivity" property of the B-spline N_m, the B-wavelet ψ_m seems to enjoy a remarkable property which we call "complete oscillation." While a precise notion of this property has yet to be described, a discussion of upper and lower bounds of the count of both strong and weak sign changes will be the main theme of this paper.

2. Identification of the Orthogonal Subspaces

From the definition of the nested sequence of spline spaces V_k^m of order m, where $k \in \mathbb{Z}$, that constitute a multiresolution analysis of L^2, it is clear that the mth order differential operator

$$D^m = \frac{d^m}{dx^m}$$

is an injection from V_k^{2m} to V_k^m for each $k \in \mathbb{Z}$. Let us introduce a closed subspace $V_{k,0}^{2m}$ of V_k^{2m}, defined by

$$V_{k,0}^{2m} = \left\{ s \in V_k^{2m} : s\left(\frac{j}{2^{k-1}}\right) = 0, j \in \mathbb{Z} \right\}. \tag{2.1}$$

Then the orthogonal complementary subspace W_k^m of V_{k+1}^m relative to V_k^m can be identified as the subspace $V_{k+1,0}^{2m}$ of V_{k+1}^{2m}, as follows.

Theorem 2.1. *Let m be an arbitrary positive integer. Then for any $k \in \mathbb{Z}$, a wavelet series g_k is in W_k^m if and only if there exists a function $s_k \in V_{k+1,0}^{2m}$ such that $s_k^{(m)}(x) = g_k(x)$. Furthermore, the differential operator D^m maps $V_{k+1,0}^{2m}$ one-one onto W_k^m.*

Proof. Without loss of generality, set $k = 0$. Let $\lambda_{2m} \in V_1^{2m}$ satisfy

$$D^m \lambda_{2m} = \psi_m. \tag{2.2}$$

Then we have

$$0 = \int_{-\infty}^{\infty} \psi_m(x) N_m(x-j) dx = [j, ..., j+m] \lambda_{2m}, \qquad j \in \mathbb{Z}.$$

Since $V_1^{2m} \cap \pi_m = \{0\}$, we may conclude that $\lambda_{2m}(j) = 0, j \in \mathbb{Z}$. Now, for any B-wavelet series

$$g(x) = \sum_{j \in \mathbb{Z}} d_j \psi_m(x-j), \qquad \{d_j\} \in l^2,$$

it follows from (2.2) that $D^m s = g$, where

$$s := \sum_{j \in \mathbb{Z}} d_j \lambda_{2m}(\cdot - j) \in V_{1,0}^{2m}.$$

On the other hand, for any function $\tilde{s} \in V_{1,0}^{2m}$, it is clear that $D^m \tilde{s} \in V_1^m$ and

$$\int_{-\infty}^{\infty} (D^m \tilde{s})(x)\, N_m(x-j)\, dx = [j, ..., j+m]\tilde{s} = 0,$$

for all $j \in \mathbb{Z}$. That is, corresponding to any $\tilde{s} \in V_{1,0}^{2m}$, we have $D^m \tilde{s} \in W_0^m$. ∎

In proving Theorem 2.1, we have seen that the spline function λ_{2m} defined in (2.2) generates the subspace $V_{1,0}^{2m}$ in the sense that the L^2-closure of the linear span of

$$\bigwedge_{2m} = \{\lambda_{2m}(\cdot - j) : j \in \mathbb{Z}\} \tag{2.3}$$

is all of $V_{1,0}^{2m}$. In the following, we will show that the set $\bigwedge_{2m}$ in (2.3) is an unconditional basis of $V_{1,0}^{2m}$.

COROLLARY 2.1 *For any positive integer m, $\bigwedge_{2m}$ is an unconditional basis of $V_{1,0}^{2m}$.*

Proof. Since $\{\psi_m(\cdot - j) : j \in \mathbb{Z}\}$ is an unconditional basis of W_0^m (cf. [4]) and the compactly supported spline function λ_{2m} is related to the B-wavelet ψ_m by $D^m \lambda_{2m} \equiv \psi_m$, it follows from Theorem 2.1 that $\bigwedge_{2m}$ is a basis of $V_{1,0}^{2m}$. To show that this basis is unconditional, it is equivalent to showing that

$$\sum_{k \in \mathbb{Z}} |\hat{\lambda}_{2m}(\omega + 2\pi k)|^2 > 0, \qquad \text{all } \omega \tag{2.4}$$

(cf. [4, 10]), since the expression in (2.4) is a 2π-periodic continuous function in ω. To verify (2.4), we note that

$$\hat{\lambda}_{2m}(\omega) = \frac{2^{-2m}}{(2m-1)!} E_{2m-1}(-z)\, \hat{N}_{2m}\left(\frac{\omega}{2}\right), \tag{2.5}$$

where $z = e^{-i\omega/2}$ and

$$E_{2m-1}(z) := (2m-1)! \sum_{j=0}^{2m-2} N_{2m}(j+1) z^j \tag{2.6}$$

is the so-called Euler–Frobenius polynomial of order $2m-1$. Hence, since E_{2m-1} never vanishes on the unit circle and

$$\sum_{j \in \mathbb{Z}} \left| \hat{N}_{2m}\left(\frac{\omega}{2} + 2\pi j\right) \right|^2 \equiv \frac{1}{(4m-1)!} \left| E_{4m-1}(z) \right|,$$

$z = e^{-i\omega/2}$, we may conclude that

$$\begin{aligned}
&\sum_{k \in \mathbb{Z}} |\hat{\lambda}_{2m}(\omega + 2\pi k)|^2 \\
&= \frac{2^{-4m}}{[(2m-1)!]^2} \left\{ |E_{2m-1}(-z)|^2 \sum_{j \in \mathbb{Z}} \left| \hat{N}_{2m}\left(\frac{\omega}{2} + 2\pi j\right) \right|^2 \right. \\
&\quad \left. + |E_{2m-1}(z)|^2 \sum_{j \in \mathbb{Z}} \left| \hat{N}_{2m}\left(\frac{\omega}{2} + \pi + 2\pi j\right) \right|^2 \right\} \\
&= \frac{2^{-4m}}{[(2m-1)!]^2 [(4m-1)!]} \{ |E_{2m-1}(-z)|^2 |E_{4m-1}(z)| \\
&\quad + |E_{2m-1}(z)|^2 |E_{4m-1}(-z)| \} > 0
\end{aligned}$$

for all $z = e^{i\omega/2}$. This completes the proof of the corollary. ∎

3. Oscillating Properties of Spline-Wavelets

Since the first order B-wavelet ψ_1 is simply the Haar function, namely,

$$\psi_1 = \chi_{(0,1/2]} - \chi_{(1/2,1]},$$

whose oscillating properties will be evident, we only restrict our attention to continuous functions f by only considering ψ_m where $m \geqslant 2$. A continuous function f is said to have an isolated sign change at x_0, if $f(x_0) = 0$ and

$$f(x_0 + t) f(x_0 - t) < 0$$

for all sufficiently small values of $t > 0$. The following notations which are somewhat standard (cf. [8, 14]) will be used for any continuous function f with compact support:

(i) supp f is the smallest closed interval outside which f vanishes identically.

(ii) $\overline{Z^c(f)}$ is the closure of the set of x such that $f(x) \neq 0$. Hence, $\overline{Z^c(f)} = \operatorname{supp} f$ iff $\overline{Z^c(f)}$ is an interval.

(iii) $S^-(f)$ is the number of strong sign changes of f, in the sense that it is the number of isolated sign changes of f that are interior points of $\overline{Z^c}(f)$.

(iv) $S^+(f)$ is the number of both strong and weak sign changes of f, in the sense that it is the count of interior isolated zeros of f with those that are not isolated sign changes counted twice.

THEOREM 3.1. *Let*

$$g(x)=\sum_{j=0}^{N} c_j\psi_m(x-j) \tag{3.1}$$

such that $c_0c_N\neq 0$ *and* $\overline{Z^c}(g)=\operatorname{supp} g$. *Then*

$$N+3m-2\leqslant S^-(g)\leqslant 2N+3m-2 \tag{3.2}$$

Furthermore, there exists a subset $\{x_1, ..., x_{N+3m-2}\}$ *of the set of isolated sign changes of g that satisfies*

$$j-m<x_j<j, \qquad j=1, ..., N+3m-2. \tag{3.3}$$

Proof. From (1.2) and (3.1), we have

$$g(x)=\sum_{k=0}^{2N+3m-2} d_k N_m(2x-k), \tag{3.4}$$

where

$$d_k=\sum_{j=0}^{N} q_{m,k-2j}c_j. \tag{3.5}$$

Hence, the upper bound in (3.2) holds, since $S^-(g)$ is bounded above by the number of strong sign changes of the coefficient sequence $\{d_k\}$ of the B-spline series representation of g (cf. [14, p. 178]). To establish the lower bound in (3.2), we first observe the simple fact that for any $f\in C^1(\mathbb{R})$ with $f(a)=f(b)=0$, where $a<b$, f' has at least one sign change in (a,b) unless f is identically zero there. Now, corresponding to the given $g\in W_0^m$ in (3.1), let us consider the spline function

$$G(x)=\sum_{j=0}^{N} c_j\lambda_{2m}(x-j) \tag{3.6}$$

in $V_{1,0}^{2m}$. Since $c_0c_N\neq 0$ and $\operatorname{supp}\psi_m=[0, 2m-1]$, we have

$$\overline{Z^c}(G)=\overline{Z^c}(g)=\operatorname{supp} g=[0, N+2m-1]. \tag{3.7}$$

Hence, it follows from the fact

$$G(k) = 0, \qquad k \in \mathbb{Z}$$
$$G^{(l)}(0) = G^{(l)}(N + 2m - 1), \qquad l = 0, ..., 2m - 2,$$

that there exists $\{x_j^l\}$, such that each x_j^l is an isolated sign change of $G^{(l)}$, where $j = 1, ..., N + 2m + l - 2$, $l = 1, ..., m$, and

$$0 < x_1^1 < 1 < x_2^1 < 2 < \cdots < N + 2m - 2 < x_{N+2m-1}^1 < N + 2m - 1$$
$$0 < x_1^2 < x_1^1 < x_2^2 < x_2^1 < \cdots < x_{N+2m-1}^1 < x_{N+2m}^2 < N + 2m - 1$$
$$\cdots\cdots\cdots$$
$$0 < x_1^m < x_1^{m-1} < x_2^m < x_2^{m-1} < \cdots < x_{N+3m-3}^{m-1} < x_{N+3m-2}^m < N + 2m - 1.$$

In particular, by setting $x_j = x_j^m$, we conclude that, for each $j = 1, ..., N + 3m - 2$, x_j is an isolated sign change of $G^{(l)} = g$, and (3.3) is satisfied. ∎

Remark 1. In view of (2.2), the spline function $G(x)$ in (3.6) has the following B-spline series representation:

$$G(x) = \sum_{k=0}^{2N+2m-2} \tilde{d}_k N_{2m}(2x - k), \tag{3.8}$$

where

$$\tilde{d}_k = (-1)^k \, 2^{-2m+1} \sum_{j=0}^{N} N_{2m}(k - 2j + 1) c_j. \tag{3.9}$$

Although an upper bound of $S^-(G)$ is given by $2N + 2m - 2$ in view of (3.8), the constraint in (3.9) of the coefficient sequence $\{\tilde{d}_k\}$ may possibly reduce this upper bound for certain **c**. Similarly, the constraint of the coefficient sequence $\{d_k\}$ in (3.5) may also reduce the upper bound of $S^-(g)$ in (3.2).

Remark 2. If we set $N = 0$ in (3.1), then (3.2) yields

$$S^-(\psi_m) = 3m - 2. \tag{3.10}$$

Hence, since ψ_m is a B-spline series with $3m - 1$ consecutive terms, it must have precisely $3m - 2$ zeros in the interior of its support $[0, 2m - 1]$, and all these zeros are simple zeros (cf. [8, 14]). The fact that ψ_m has minimum support has already been established in [4]. We also note that by the symmetry or antisymmetry of ψ_m, depending on even or odd m, these simple zeros of ψ_m (which are isolated sign changes) are symmetric with respect to the midpoint $x = m - 1/2$ of supp ψ_m. To be more specific, let

$$X_m = \{x_{m,1}, ..., x_{m,3m-2}\},$$

where $0 < x_{m,1} < \cdots < x_{m,3m-2} < 2m-1$, be the zero set of ψ_m in $(0, 2m-1)$. Then we have

$$x_j = 2m - 1 - x_{3m-j-1}, \qquad j = 1, ..., 3m-2.$$

These zeros also satisfy (3.3) for $N=0$. A listing of the zero set X_m of ψ_m for $m = 2, 3, 4$ is shown in Table I. It is noted that X_m satisfies the stricter constraint

$$\max\left(\frac{j}{2}, j-m\right) < x_{m,j} < \min\left(\frac{j+m-1}{2}, j\right), \tag{3.11}$$

for $m = 2, 3, 4$. In fact, as pointed out by Goodman [7], this observation is valid for all m, as follows.

Proposition 3.1. *Let $m \geqslant 2$ be any positive integer. Then the zeros $x_{m,1}, ..., x_{m,3m-2}$ of the mth order B-wavelet ψ_m satisfy the inequalities in (3.11).*

Proof. In view of (3.3), it is sufficient to verify

$$\frac{j}{2} < x_{m,j} < \frac{j+m-1}{2}.$$

Furthermore, by changing $2x$ to x, it is sufficient to prove that for any spline function

$$f(x) = \sum_{k=0}^{N} c_k N_m(x-k)$$

TABLE I

Zeros of ψ_m

	Zeros		Zeros
ψ_2	$x_{2,1} = 0.574218$	ψ_4	$x_{4,1} = 0.625000$
	$x_{2,2} = 1.187500$		$x_{4,2} = 1.312500$
	$x_{2,3} = 1.812500$		$x_{4,3} = 1.968750$
	$x_{2,4} = 2.425782$		$x_{4,4} = 2.585938$
ψ_3	$x_{3,1} = 0.605562$		$x_{4,5} = 3.195312$
	$x_{3,2} = 1.269625$		$x_{4,6} = 3.804688$
	$x_{3,3} = 1.898438$		$x_{4,7} = 4.414062$
	$x_{3,4} = 2.500000$		$x_{4,8} = 5.031250$
	$x_{3,5} = 3.101562$		$x_{4,9} = 5.687500$
	$x_{3,6} = 3.731375$		$x_{4,10} = 6.375000$
	$x_{3,7} = 4.394438$		

with isolated zeros $t_1 < \cdots < t_r$, we have

$$i < t_i < i + N - r + m - 1,$$

for all $i = 1, ..., r$. By symmetry, we only need to establish the lower bound. Assume, on the contrary, that the lower bound fails for $i = j$, say. Then the spline function

$$\tilde{f}(x) = \sum_{k=0}^{j} c_k N_m(x-k),$$

being identical to $f(x)$ on $(-\infty, j]$, satisfies

$$\tilde{f}(t_i) = 0, \qquad i = 1, ..., j,$$

where $t_j \leqslant j$. Since it is clear from its formulation that $\tilde{f}$ has at most $j-1$ isolated zeros in the open interval $(0, j)$, we have

$$t_j = j \qquad \text{and}$$

$$\tilde{f}(x) = 0, \qquad j \leqslant x \leqslant j+1.$$

This implies that

$$c_{j-m+1} = \cdots = c_j = 0$$

by the local linear independence of B-splines; and consequently, $\tilde{f}$ has at most $j - m \leqslant j - 2$ isolated zeros. This is a contradiction. ∎

Remark 3. It follows from Proposition 3.1 and an application of the Schoenberg–Whitney theorem, that even a stronger version of uniqueness of ψ_m holds; namely, if $g \in V_1^m$ (which may not even be in the subspace W_0^m of V_1^m) satisfies

$$\operatorname{supp} g = [0, 2m-1],$$

such that each $x_{m,j}$, $j = 1, ..., 3m-2$, is an isolated sign change of g, then g is a constant multiple of ψ_m. This observation is in the same spirit as signal recovery from its zero-crossings to be discussed in the next section.

We now turn to the study of the stronger count $S^+(g)$ of both strong and weak sign changes. The same standard notation for counting sign changes of finite sequences of real numbers

$$\mathbf{a} = \{a_k\}, \qquad k = 0, ..., n, \tag{3.12}$$

of length $n+1$, will be used; namely, $S^-(\mathbf{a})$ is the count of actual (or strong) sign changes in $\{a_0, ..., a_n\}$ after all the zero terms are deleted, and

$S^+(\mathbf{a})$ is the count if each zero term is changed to $+1$ or -1 so as to maximize the count. Hence, the following relation can be easily verified (cf. [14, p. 25]).

LEMMA 3.1. *Let* **a** *be a real sequence of lenth* $n+1$ *as given in* (3.12), *and set*

$$\mathbf{a}^* = \{(-1)^k a_k\}, \qquad k = 0, ..., n. \tag{3.13}$$

Then

$$S^+(\mathbf{a}) + S^-(\mathbf{a}^*) \geqslant n. \tag{3.14}$$

For linear wavelets, we have the following result concerning $S^+(g)$.

THEOREM 3.2. *Let* g *be the wavelet series given in* (3.1) *with* $m=2$, $c_0 c_N \neq 0$, *and* $\overline{Z^c(g)} = \operatorname{supp} g$. *Then*

$$2N + 4 - S^-(\mathbf{c}) \leqslant S^+(g) \leqslant 2N + 4. \tag{3.15}$$

In particular, if the coefficient sequence $\mathbf{c} = \{c_j\}$ *in* (3.1) *is of one sign, i.e.,* $S^-(\mathbf{c}) = 0$, *then*

$$S^+(g) = 2N + 4. \tag{3.16}$$

Remark 4. The upper bound in (3.15) is a consequence of the spline series expansion (3.14) with $m=2$ and a well-known result on zero count (cf. [8, p. 233; 14]).

Remark 5. It is tempting to generalize (3.15) to wavelet series of arbitrary order. A plausible generalization would be

$$2N + 3m - 2 - d(m) - S^-(\mathbf{c}) \leqslant S^+(g) \leqslant 2N + 3m - 2 - d(m), \tag{3.17}$$

for some non-negative integer $d(m)$, with $d(2) = 0$. In this regard, it was already pointed out by T. N. T. Goodman (cf. [7]) that $d(4) > 0$. Perhaps (3.17) would hold for only a subclass of wavelet series, where the variation of the coefficient sequence **c** is subject to certain constraint. That $d(m)$ should be nonzero is somewhat reasonable due to restriction of the coefficient sequence $\{d_k\}$, governed by (3.5), of the *B*-spline series (3.4). With the validity of (3.17), the property of "complete oscillation" of wavelet series, which is possessed by linear wavelets as shown in (3.16), could be generalized to spline-wavelets of higher orders.

We now turn to the proof of Theorem 3.2.

Proof. We need the values of $\psi_2(j/2), j \in \mathbb{Z}$, which can be easily computed by using (1.2) and (1.3). These values are given as

$$
\begin{aligned}
&\psi_2\left(\frac{1}{2}\right) = \psi\left(\frac{5}{2}\right) = \frac{1}{12} \\
&\psi_2\left(\frac{2}{2}\right) = \psi_2\left(\frac{4}{2}\right) = -\frac{1}{2} \\
&\psi_2\left(\frac{3}{2}\right) = \frac{5}{6} \\
&\psi_2\left(\frac{j}{2}\right) = 0 \qquad \text{for} \quad j \leqslant 0 \quad \text{or} \quad j \geqslant 6.
\end{aligned}
\tag{3.18}
$$

With the values in (3.18), we form the $(2N+5) \times (N+1)$ matrix

$$
B := [b_{i,j}] := \left[(-1)^{i-2j+1} \psi_2\left(\frac{i-2j+2}{2}\right) \right] \tag{3.19}
$$

$$
= \frac{1}{12} \begin{bmatrix}
1 & & & & \\
6 & & & & \\
10 & 1 & & & \\
6 & 6 & & & \\
1 & 10 & 1 & & \\
 & 6 & 6 & & \\
 & 1 & 10 & & \\
 & & 6 & \cdots & 1 \\
 & & 1 & & 6 \\
 & & & & 10 \\
 & & & & 6 \\
 & & & & 1
\end{bmatrix},
$$

where the entries not entered are all zeros. It is easy to show that B is of class TP_{N+1} (cf. [8, p. 12] for the definition). By setting

$$
\begin{aligned}
\mathbf{v}_g &= \left(g\left(\frac{1}{2}\right), g\left(\frac{2}{2}\right), \ldots, g\left(\frac{2N+5}{2}\right) \right) \\
\mathbf{v}_g^* &= \left(g\left(\frac{1}{2}\right), -g\left(\frac{2}{2}\right), g\left(\frac{3}{2}\right), \ldots, (-1)^{2N+4} g\left(\frac{2N+5}{2}\right) \right),
\end{aligned}
\tag{3.20}
$$

the wavelet series (3.1) at $x = 1/2, ..., 2N + 5$, with $m = 2$, can be written in matrix form as

$$\mathbf{v}_g^* = B\mathbf{c}.$$

By a well-known result on total positivity (cf. [8, p. 223]), we have

$$S^-(\mathbf{v}_g^*) \leqslant \min(N, S^-(\mathbf{c})) = S^-(\mathbf{c}),$$

so that an application of Lemma 3.1 yields

$$\begin{aligned} S^+(\mathbf{v}_g) &\geqslant 2N + 4 - S^-(\mathbf{v}_g^*) \\ &\geqslant 2N + 4 - S^-(\mathbf{c}). \end{aligned}$$

This establishes the lower bound in (3.15) and completes the proof of the theorem. ▮

4. Characterization of Signals in Terms of Zero-Crossings

It is well known that, under certain conditions, a bandpass band-limited signal can be completely recovered from its zero-crossings (or sign changes) (cf. [9]]). The reason for this to possibly hold is that under certain conditions such as being in $L^2(\mathbb{R})$ and satisfying a given normalization condition, an entire function of exponential type τ is completely characterized by its real zeros whose distribution is governed by the type τ (cf. [1]). Since a spline-wavelet series is piecewise analytic, it is hoped that some analogous results are still plausible. In the following, we only give a very modest contribution to this problem in order to initiate this important direction of research.

THEOREM 4.1. *Let*

$$f(x) = \sum_{j=0}^{N} d_j \psi_2(x - j);$$

$$g(x) = \sum_{j=0}^{N} c_j \psi_2(x - j)$$

be two linear spline-wavelet series with $\overline{Z^c}(f) = \overline{Z^c}(g) = [0, N + 3]$ *such that* $S^-(\mathbf{c}) = 0$ *and g has only simple zeros. Then if f and g have the same zeros, f must be a constant multiple of g.*

Proof. By assumption, we have $d_0 d_N \neq 0$ and $c_0 c_N \neq 0$. Choose c such that $c_0 = cd_0$. Then

$$(g - cf)(x) = \sum_{j=1}^{N} (c_j - cd_j)\, \psi_2(x - j).$$

Suppose that $g - cf$ is not identically zero. Then we may assume, without loss of generality, $\overline{Z^c}(g - cf) = [1, N+3]$. Since $(g - cf) \in V_1^2$, an application of the Budan–Fourier theorem (cf. [14, p. 163]) affirms that the number of zeros $Z(g - cf)$, counting multiplicities, of $g - cf$ in the open interval $(1, N+3)$ does not exceed $2N+2$. That is, we have

$$Z(g - cf) \leqslant 2N + 2. \tag{4.1}$$

On the other hand, since the restriction of g to $(0, 1]$ is a constant multiple of ψ_2, it has only a simple zero in $(0, 1]$. Hence, it follows from Theorem 3.2 that

$$S^+(g)|_{(1, N+3)} = S^+(g)|_{(0, N+3)} - 1 \geqslant 2N + 4 - S^-(\mathbf{c}) - 1 = 2N + 3. \tag{4.2}$$

In addition, since g has only simple zeros and f has the same zeros as g, we have

$$Z(g)|_{(1, N+3)} \leqslant Z(g - cf). \tag{4.3}$$

Therefore, from (4.1), (4.2), and (4,3), we obtain

$$2N + 3 \leqslant S^+(g)|_{(1, N+3)} \leqslant Z(g)|_{(1, N+3)} \leqslant Z(g - cf) \leqslant 2N + 2,$$

which is absurd. This completes the proof of the theorem. ∎

5. Acknowledgments and Final Remarks

We do not have a good conjecture for the value of $d(m)$ in (3.16) other than the known value $d(2) = 0$ as given in Theorem 3.2. The original guess of $d(m) = 0$ was incorrect as pointed out by T. N. T. Goodman [7], to whom we are very grateful. The consideration of zero-crossings was motivated by a very stimulating discussion with S. Mallat on his recent work on "wavelet maxima" (cf. [11]). We would also like to point out a very interesting recent work on non-uniform spline-wavelets by Buhman and Micchelli [2] which was available to us after the completion of this paper, where (3.2) was also established independently by a different approach. Finally, we thank the referees for their valuable comments that improved the presentation of the paper.

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